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Luca Bellotti

# **What is a model of axiomatic set theory?**



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## PREFACE

This is a work in the philosophy of mathematics, about some philosophical problems regarding set theory, in particular the difficult conceptual question of semantics for set theory. Specifically, some aspects of the classical philosophical question concerning the notion of *model of axiomatic set theory* are discussed.

The set-theoretic nature of the usual semantics of set theory raises a problem of conceptual (as opposed to mathematical) circularity (Chapter 1). A recourse to an intuitive semantics (possibly in terms of the iterative conception of set) is apparently necessary, and a realist philosophy of mathematics seems its best justification, taking for granted a well-determined reality of which set-theoretic statements are true. But the realist's need for a univocal interpretation of set theory against the proliferation of unintended models seems doomed to remain unfulfilled. In view of this question I discuss (Ch. 2) the Skolem 'paradox'. The dispute between Skolemites and Antiskolemites and the notion of informal mathematics are examined from a point of view which I hope to be rather unusual. Then (Ch. 3), the natural second-order alternative to the 'weak' first-order formal systems is examined, trying to give an idea of the entangled relationship between second-order logic and set theory. Afterwards (Ch. 4) I make an attempt at the description of a very tangled affair, namely the relationship between the syntactic metatheory and the semantical metatheory of axiomatic set theory. It turns out that it is difficult to save a notion of syntactic metatheory immune against the instability in the notion of finiteness deriving from the nonstandard interpretability of arithmetic. I conclude my work (Ch. 5) considering the question of the consistency of axiomatic set theory, touching some epistemological aspects of the problem. In an appendix, I give a schematic survey (necessary as a reference) of large cardinals and inner models, with a glance at the role of formalization in the justification of higher infinity principles.

My unitary standpoint emerges in the course of my arguments and cannot be summarized here. However, as a first approximation, I can say what follows. Platonist realism, which is the apparent solution to all the above problems, leaves one in even deeper trouble. My position, *lato sensu* a Neo-Kantian one (closer to 'Marburger' Neo-Kantianism than to Kant), is different: I maintain that the crucial feature of mathematical 'reality' is the objectivity of concepts, concepts which, however, do not subsist as ontologically given substances, nor as mental or linguistic constructs, but as something which by its own essence *holds* in an autonomous realm of pure validity. This claim is argued for by looking at the actual practice of the mathematical study of models of set theory, which is taken as a datum for philosophical reflection.

This book could, I hope, give its contribution to the literature in the philosophy of mathematics because it discusses some classical problems in the field and examines some important writings on them in the light of a unitary overall point of view, in a broad sense Neo-Kantian, which has been rather

unusual in the last decades, at least in the literature in the analytic tradition in philosophy.

This work has a primary audience of all those who are interested in the history and the philosophy of set theory. However, the problems discussed, taking set theory as the object of a case study, have broader scope and interest, and could perhaps be interesting also for philosophers not familiar with the current literature in the philosophy of set theory, provided they have a basic knowledge in foundations and an interest in the philosophical problems discussed, or in a general approach to the formal sciences which is different from the ones customary in the analytic tradition.

The first idea of this essay dates back to more than ten years ago, and a first draft to a few years later, but for various reasons its publication has been delayed for many years. Of course, some of the topics treated here have been much discussed in the literature in the meantime, and some parts of the book would perhaps need, in view of this, a complete rewriting (as opposed to minor modifications). For various reasons, I was not able to do any substantial rewriting; in my opinion, however, the basic problems and the main approaches available are still the same, so I hope there is still some value in the arguments I make.

Slightly different versions of some parts of this book have appeared before in some journals: *Epistemologia* (parts of Chapters 1 and 5: respectively in Vol. 36, 2013, in press, and Vol. 29, 2006, pp. 41-60; publisher: Franco Angeli); *Theoria* (most of Ch. 2: in Vol. 72, 2006, pp. 177-212; publisher: Wiley-Blackwell); *Synthese* (most of Ch. 4: in Vol. 154, 2007, pp. 199-229; publisher: Springer); *L&PS* (part of Ch. 4: in Vol. 9, 2011, electronic; publisher: University of Trieste). I thank the publishers for allowing the right to republish the parts of the texts which have remained unchanged. Finally, I wish to thank Tatiana Arrigoni, Marco Forti, Paolo Mancosu and especially Enrico Moriconi, for their valuable comments on previous drafts and their helpfulness.

L. B.

*La Spezia, Italy*  
*October 2012*

*Since, then, we are in a difficulty, please to tell us what you mean, when you speak of being; for there can be no doubt that you always from the first understood your own meaning, whereas we once thought that we understood you, but now we are in a great strait.*

PLATO

*The eternal silence of these infinite spaces frightens me.*

BLAISE PASCAL

*It would be truly sad if this wave of success should succeed in totally dismissing all philosophical concern over CH and similar questions as inconsequential. Of course, good mathematics is beautiful, while most philosophical discussion is barren and certainly not beautiful.*

PAUL J. COHEN